J. Fluid Mech. (1988), *vol.* 186, *pp.* 559–581
Printed in Great Britain

On perturbations of Jeffery–Hamel flow

By W. H. H. BANKS, P. G. DRAZIN AND M. B. ZATURSKA
School of Mathematics, University of Bristol, Bristol BS8 1TW, UK

(Received 15 January 1987)

We examine various perturbations of Jeffery–Hamel flows, the exact solutions of the Navier–Stokes equations governing the steady two-dimensional motions of an incompressible viscous fluid from a line source at the intersection of two rigid plane walls. First a pitchfork bifurcation of the Jeffery–Hamel flows themselves is described by perturbation theory. This description is then used as a basis to investigate the spatial development of arbitrary small steady two-dimensional perturbations of a Jeffery–Hamel flow; both linear and weakly nonlinear perturbations are treated for plane and nearly plane walls. It is found that there is strong interaction of the disturbances up- and downstream if the angle between the planes exceeds a critical value $2\alpha_2$, which depends on the value of the Reynolds number. Finally, the problem of linear temporal stability of Jeffery–Hamel flows is broached and again the importance of specifying conditions up- and downstream is revealed. All these results are used to interpret the development of flow along a channel with walls of small curvature. Fraenkel's (1962) approximation of channel flow locally by Jeffery–Hamel flows is supported with the added proviso that the angle between the two walls at each station is less than $2\alpha_2$.

1. Introduction

Jeffery–Hamel flows are a family of exact solutions of the Navier–Stokes equations for steady two-dimensional flow of an incompressible viscous fluid from a line source at the intersection of two rigid planes. Although they were discovered over seventy years ago by Jeffery (1915) and Hamel (1916), and have been described in many textbooks, they are not widely understood. This is surely because, depending on two dimensionless parameters, there is a multiplicity of solutions with a richer structure than most other similarity solutions.

They are also more important than merely one family in the mathematical menagerie of exact solutions of the Navier–Stokes equations because they may be generalized and used in a variety of contexts. In particular, they may be used to approximate locally the steady flow in a two-dimensional channel with walls of small curvature (Fraenkel 1962). To see this, consider steady flow along a channel in the (x, z)-plane which has walls with equations $z = f(x)$ and $z = -g(x)$. Suppose that there is a given steady flux Q in the positive x-direction. Now the equations of the walls may be equivalently expressed as $\theta = \alpha(x)$ and $\theta = -\beta(x)$, where $\tan\alpha(x) = f'(x)$, $\tan\beta(x) = g'(x)$, and θ is defined as the angle the tangent to a wall at the station x makes with the positive x-axis. Then it can be seen in figure 1 that the channel is *locally* like the configuration of a Jeffery–Hamel flow if α and β vary slowly, i.e. if the curvature of the walls is small. Watson (Fraenkel 1963, p. 407) noted that this local approximation holds whether the channel is symmetric, and so $\beta(x) = \alpha(x)$ for all x, or not. Recently Sobey & Drazin (1986) have not only used Jeffery–Hamel flows to

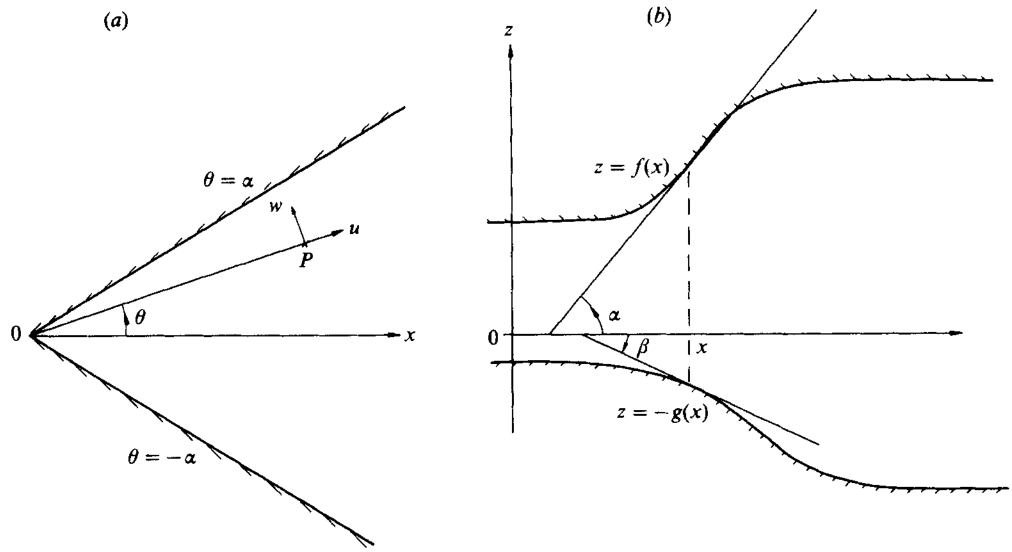

FIGURE 1. (a) Sketch of configuration of Jeffery–Hamel flow. (b) Sketch of two-dimensional channel approximated locally at station x by configuration of Jeffery–Hamel flow.

describe channel flow but also simplified the presentation of Jeffery–Hamel flows by use of bifurcation diagrams and suggested that each flow with radial velocity distribution of both signs is unstable. This led them to question the validity of using Jeffery–Hamel flows to approximate flow in a channel with walls of small curvature wherever there is no appropriate Jeffery–Hamel flow with radial velocity of only one sign.

Although we assume that the curvature of the walls is small, we in general do not assume that the angles α and β are small. This is in contrast to several papers by Eagles and co-authors (see, e.g., Georgiou & Eagles 1985), who have exploited the use of parallel flow as a first approximation to elucidate Jeffery–Hamel flows, channel flows and their instabilities. However, Allmen & Eagles (1984) solved an interesting partial-differential eigenvalue problem in a finite wedge of given non-zero angle; they calculated some forced linear modes of given frequency to compare with results of spatially growing modes in a slowly diverging channel.

In this paper we shall study various small perturbations of Jeffery–Hamel flows in detail. First we study steady perturbations to find the spatial growth of small disturbances. Perturbations among the Jeffery–Hamel flows themselves are given in §2. A pitchfork bifurcation is analysed. At the same time the apparatus of operator theory is built to solve that and subsequent problems. Small steady perturbations of Jeffery–Hamel flows which are not of the Jeffery–Hamel similarity form are treated in §3. The linearized problem is soluble by use of normal modes, the variables being separable so that an ordinary differential eigenvalue problem arises. The special case of this problem for zero Reynolds number was solved by Lugt & Schwiderski (1965) and interpreted by Sternberg & Koiter (1958). We solve also another special case, the one that marks the marginal stability of the spatially growing steady modes. This special case is then used in §4 as a first approximation for the weakly nonlinear spatial growth of steady modes in a channel with walls of small curvature. The problem of linear temporal stability of Jeffery–Hamel flows is broached in §5, and it is shown that boundary conditions at infinity and at the line source affect the stability characteristics.

2. Jeffery–Hamel flows and their bifurcations

The stream function ψ of two-dimensional flow of an incompressible fluid of kinematic viscosity ν satisfies the vorticity equation,

$$\nu\nabla^4\psi + \frac{1}{r}\frac{\partial(\psi, \nabla^2\psi)}{\partial(r,\theta)} = \frac{\partial\nabla^2\psi}{\partial t}, \tag{2.1a}$$

where the Laplacian in cylindrical polar coordinates is $\nabla^2 = \partial^2/\partial r^2 + \partial/r\,\partial r + \partial^2/r^2\,\partial\theta^2$. We shall suppose that the flow is driven by a given steady volume flux Q between the two rigid walls with equations $\theta = \pm\alpha$. Then the boundary conditions of no slip and impermeability are

$$\psi = \pm\tfrac{1}{2}Q, \quad \frac{\partial\psi}{\partial\theta} = 0 \quad \text{at } \theta = \pm\alpha. \tag{2.1b}$$

First we shall consider steady Jeffery–Hamel (called JH hereinafter) flows with stream function $\psi(r,\theta) = \frac{1}{2}QG(y,\alpha,R)$, where $R = Q/2\nu$ is the Reynolds number and $y = \theta/\alpha$. The system (2.1) becomes

$$G_{yyyy} + 4\alpha^2 G_{yy} + 2\alpha R G_y G_{yy} = 0 \tag{2.2a}$$

and

$$G = \pm 1, \quad G_y = 0 \quad \text{at } y = \pm 1. \tag{2.2b}$$

Jeffery (1915) and Hamel (1916) themselves showed that solutions of the problem (2.2) may be expressed explicitly in terms of Jacobian elliptic functions. Since then JH flows have become well known through the work of many authors. All such flows have been classified by Fraenkel (1962) into types I, II_n, III_n, IV_n or V_n for $n = 1, 2, \ldots$. A few cases are especially simple and useful. Plane Poiseuille flow arises for solutions of types I and III_1 as $\alpha \to 0$ for fixed R. Stokes flow arises for solutions of types I, II_n and III_n, with

$$G(y,\alpha,R) \to \frac{\sin 2\alpha y - 2\alpha y\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha} \quad \text{as } R \to 0$$

for fixed α (which is not a root of $\tan 2\alpha = 2\alpha$). All solutions intermediate between types II_1, II_2, IV_1 and V_1 are given by Fraenkel (1962) on what he called boundary $\mathscr{B}_2$ with equation $\alpha = \alpha_2(R)$, where α_2 is expressed in simple terms of complete elliptic integrals.

Sobey & Drazin (1986) recognized that a subcritical pitchfork bifurcation of the JH solutions occurs on the boundary $\mathscr{B}_2$. To examine the nature of this bifurcation by the use of perturbation theory, define $y_2 = \theta/\alpha_2$ and $\epsilon = \alpha - \alpha_2$ so that (2.2) becomes

$$G^{\mathrm{iv}} + 4\alpha_2^2 G'' + 2\alpha_2 R G' G'' = 0 \tag{2.3a}$$

and

$$G = \pm 1, \quad G' = 0 \quad \text{at } y_2 = \pm\left(1 + \frac{\epsilon}{\alpha_2}\right), \tag{2.3b}$$

where a prime denotes differentiation with respect to y_2. Then expand

$$G = G_0 + \epsilon^{\frac{1}{2}} G_{\frac{1}{2}} + \epsilon G_1 + \epsilon^{\frac{3}{2}} G_{\frac{3}{2}} + \ldots \quad \text{as } \epsilon \to 0 \tag{2.4}$$

for fixed R, and substitute the expansion into (2.3). First we equate coefficients of terms in ϵ^0 and find

$$G_0^{\mathrm{iv}} + 4\alpha_2^2 G_0'' + 2\alpha_2 R G_0' G_0'' = 0 \tag{2.5a}$$

and

$$G_0 = \pm 1, \quad G_0' = 0 \quad \text{at } y_2 = \pm 1. \tag{2.5b}$$

This gives the JH solution G_0 on boundary $\mathscr{B}_2$, which marks the bifurcation of solutions of types II_1, II_2, IV_1 and V_1. The solution G_0, representing a symmetric flow, is an odd function of y_2, and is also such that $G_0'' = 0$ at $y_2 = \pm 1$.

Next we equate coefficients of $\epsilon^{\frac{1}{2}}$ in (2.3) and find

$$\mathrm{L}G_{\frac{1}{2}} = 0 \tag{2.6a}$$

and

$$\mathbf{B}G_{\frac{1}{2}} = \mathbf{0}, \tag{2.6b}$$

where we define the linear operators L and $\mathbf{B}$ so that

$$\mathrm{L}u = u^{\mathrm{iv}} + 4\alpha_2^2 u'' + 2\alpha_2 R(G_0' u')' \tag{2.7}$$

and

$$\mathbf{B}u = [u(-1), u(1), u'(-1), u'(1)]^{\mathrm{T}} \tag{2.8}$$

for all functions $u \in S$, and

$$S = \left\{ u \,\middle|\, \int_{-1}^{1} (u^{\mathrm{iv}})^2 \, \mathrm{d}y_2 < \infty \right\}.$$

Now $\mathrm{L}G_0' = 0$ (a result which does not depend on the special value of α_2) and $\mathbf{B}G_0' = \mathbf{0}$ (a result which does), so the general solution of (2.6) is

$$G_{\frac{1}{2}} = aG_0' \tag{2.6c}$$

for an arbitrary constant a. We shall evaluate a later when solving the problem for $G_{\frac{3}{2}}$.

It is convenient now to digress, in order to prepare to solve a few inhomogeneous linear problems of the form

$$\mathrm{L}u = h', \quad \mathbf{B}u = [b_1, b_2, b_3, b_4]^{\mathrm{T}} \tag{2.9}$$

for given functions $h(y_2)$ and constants b_1, b_2, b_3, b_4. Fraenkel (1963, §5) derived such problems in a closely related context and showed how to solve them. It helps first to define an inner product by

$$(u, v) = \int_{-1}^{1} uv \, \mathrm{d}y_2, \tag{2.10}$$

so that the generalized Lagrange identity,

$$(\mathrm{L}u, v) - (u, \mathrm{L}v) = [I(u, v)]_{-1}^{1} \tag{2.11}$$

follows for all $u, v \in S$ on integration by parts, where I is defined by

$$I(u, v) = u'''v - u''v' + u'v'' - uv''' + (4\alpha_2^2 + 2\alpha_2 RG_0')(u'v - uv'). \tag{2.12}$$

The Lagrange identity for $v = G_0'$ gives the solvability condition for the existence of the solution u of (2.9),

$$(h', G_0') = [I(u, G_0')]_{-1}^{1},$$

i.e.

$$\int_{-1}^{1} G_0'' h \, \mathrm{d}y_2 = (b_4 - b_3) G_0'''(1), \tag{2.13}$$

on integration by parts and use of the boundary conditions satisfied by u and G_0'. Further, $\mathrm{L}u = (\mathrm{M}u')'$, where M is the linear differential operator defined by $\mathrm{M}v = v'' + (4\alpha_2^2 + 2\alpha_2 RG_0')v$ for all $u, v \in S$. Now $\mathrm{M}G_0'' = 0$ and

$$\mathrm{M}\left[G_0''(y_2) \int^{y_2} \{G_0''(y)\}^{-2} \mathrm{d}y \right] = 0,$$

so if condition (2.13) is satisfied we may solve (2.9) (in terms of quadratures) by variation of parameters, although in practice it is usually easier to find u by direct numerical integration of (2.9). Here ends the digression.

Next we equate coefficients of ϵ in (2.3) and find

$$\mathbf{L}G_1 = -a^2\alpha_2 R\{(G_0'')^2\}' \tag{2.14a}$$

and

$$\mathbf{B}G_1 = \mathbf{0}. \tag{2.14b}$$

The solvability condition (2.13) for (2.14) is 'automatically' satisfied because G_0 is an odd function of y_2. Therefore there is a solution of the form

$$G_1 = a^2 F_1, \tag{2.14c}$$

where F_1 is defined as the unique odd solution of the problem

$$\mathbf{L}F_1 = -\alpha_2 R\{(G_0'')^2\}' \tag{2.15a}$$

and

$$\mathbf{B}F_1 = \mathbf{0}. \tag{2.15b}$$

The particular integral $\frac{1}{2}G_0''$ of (2.15*a*) and the known complementary function may be used to give F_1 by quadratures and thence in explicit terms of Jacobian elliptic functions, but we have found it easier to obtain F_1 by numerical integration of the system (2.15). Details are given later in this section.

Coefficients of $\epsilon^{\frac{3}{2}}$ in problem (2.3) give

$$\mathbf{L}G_{\frac{3}{2}} = -2a^3\alpha_2 R(G_0'' F_1')' \tag{2.16a}$$

and

$$\mathbf{B}G_{\frac{3}{2}} = \frac{a}{\alpha_2}[0, 0, G_0'''(-1), -G_0'''(1)]^{\mathrm{T}}. \tag{2.16b}$$

Now the solvability condition (2.13) for system (2.16) gives

$$-a^3\alpha_2 R\int_{-1}^{1} F_1'(G_0'')^2\,\mathrm{d}y_2 = \frac{a\{G_0'''(1)\}^2}{\alpha_2},$$

i.e.

$$a = 0 \quad \text{or } a^2 = -\frac{\{G_0'''(1)\}^2}{\alpha_2^2 R\int_{-1}^{1} F_1'(G_0'')^2\mathrm{d}y_2}. \tag{2.17}$$

It may be recognized that the non-zero roots $\pm a$ of (2.17) give the asymmetric JH solutions G of types IV_1 and V_1 for small ϵ. We anticipate that the integral in the denominator is positive for all R because the asymmetric solutions are known to be real only for $\alpha < \alpha_2$, i.e. for $\epsilon < 0$. Thus formally $\epsilon^{\frac{1}{2}}$ and a are purely imaginary but the product $\epsilon^{\frac{1}{2}}a$, and therefore G, are real. For $R = 0$, so that $\alpha_2 = \frac{1}{2}\pi$, we find that $G_0 = y + \pi^{-1}\sin\pi y$ and so, for $0 < R \ll 1$, $F_1 = -\frac{1}{24}R\,(\sin 2\pi y + 2\sin\pi y)$ and from (2.17) we obtain $a^2 \sim -96/(\pi R^2)$ as $R \to 0$. Numerical integration of (2.5) and (2.15) has been carried out and we find that, for example, if $R = 20$ and $\alpha = \alpha_2$ then $a^2 = -13.080$.

The root $a = 0$ of (2.17) gives the symmetric JH solutions of types II_1 and II_2 as $\epsilon \to 0$. In this case $G_{m+\frac{1}{2}}(y_2) = 0$ for $m = 0, 1, \ldots$ and $G_1(y_2) = 0$ for all y_2. Therefore we need to proceed further to find the leading term which modifies the JH solution for small ϵ, i.e. to find G_2. Accordingly, equate coefficients of ϵ^2 in (2.3) for the case $a = 0$ to find

$$\mathbf{L}G_2 = 0 \tag{2.18a}$$

and

$$\mathbf{B}G_2 = -(2\alpha_2^2)^{-1}[0, 0, G_0'''(-1), G_0'''(1)]^{\mathrm{T}}. \tag{2.18b}$$

The solvability condition for (2.18) is satisfied because G_0 is an odd function. Again, we may regard the odd solution G_2 as known in terms of quadratures or by direct numerical integration.

3. Spatially developing steady modes

Here we shall describe steady small perturbations of a Jeffery–Hamel flow, showing how they develop up- and downstream. This will lead to consideration of boundary-value problems with the domain of flow $-\alpha \leqslant \theta \leqslant \alpha$, $r_1 \leqslant r \leqslant r_2$. Accordingly, take a JH solution as the basic flow, substitute $\psi = \frac{1}{2}Q(G+\delta\psi')$ into (2.1), and linearize the system for small perturbations to find

$$\nabla^4\psi' + \frac{R}{r}\frac{\partial(G, \nabla^2\psi')}{\partial(r,\theta)} + \frac{R}{r}\frac{\partial(\psi', \nabla^2 G)}{\partial(r,\theta)} = 0 \tag{3.1a}$$

and

$$\psi' = 0, \quad \frac{\partial\psi'}{\partial\theta} = 0 \quad \text{at } \theta = \pm\alpha \tag{3.1b}$$

in the limit as the parameter $\delta \to 0$. It can be seen that we may separate the variables to solve this linearized system, taking 'normal modes' of the form

$$\psi'(r,\theta,\alpha,R) = r^{\lambda}\hat{\psi}(y,\alpha,R). \tag{3.2}$$

Then (3.1) becomes

$$\hat{\psi}_{yyyy} + \alpha^2\{\lambda^2 + (\lambda-2)^2\}\hat{\psi}_{yy} + \alpha^4\lambda^2(\lambda-2)^2\hat{\psi}$$
$$-\alpha R(\lambda-2)G_y(\hat{\psi}_{yy} + \alpha^2\lambda^2\hat{\psi}) + \alpha R\lambda G_{yyy}\hat{\psi} + 2\alpha R G_{yy}\hat{\psi}_y = 0 \tag{3.3a}$$

and

$$\hat{\psi} = 0, \quad \hat{\psi}_y = 0 \quad \text{at } y = \pm 1. \tag{3.3b}$$

The eigenvalue problem (3.3) may be solved directly by a numerical method such as 'shooting' to determine λ and $\hat{\psi}$ in terms of α and R because the functions G are known. However, we may solve the problem analytically in two special cases of importance.

3.1. *The case $R = 0$*

The problem (3.1) when $R = 0$ is well known (after Dean & Montagnon 1949), because it reduces to the solution of the biharmonic equation and represents problems of plane elasticity as well as Stokes flow in a wedge. The eigensolutions may be of two types according as $\hat{\psi}$ is an even or odd eigenfunction.

(i) *Even eigenfunctions*:

$$\hat{\psi}(y) = \frac{\cos\alpha\lambda y}{\cos\alpha\lambda} - \frac{\cos\alpha(\lambda-2)y}{\cos\alpha(\lambda-2)}, \tag{3.4a}$$

where λ is a zero of $V(-\lambda)$ and V is defined by

$$V(p) = (p+1)\sin 2\alpha + \sin 2\alpha(p+1) \quad \text{for all } p. \tag{3.4b}$$

We denote this family of eigenvalues by l_i where $i = \pm 1, \pm 2, \ldots$; positive subscripts will denote eigenvalues whose real parts are greater than or equal to $\frac{3}{2}$, and negative subscripts for those less than or equal to $\frac{1}{2}$.

(ii) *Odd eigenfunctions*:

$$\hat{\psi}(y) = \frac{\sin \alpha\lambda y}{\lambda \cos \alpha\lambda} - \frac{\sin \alpha(\lambda-2)y}{(\lambda-2)\cos \alpha(\lambda-2)}, \tag{3.5a}$$

where λ is a zero of $W(-\lambda)$ and W is defined by

$$W(p) = (p+1)\sin 2\alpha - \sin 2\alpha(p+1) \quad \text{for all } p. \tag{3.5b}$$

This family of eigenvalues is denoted by m_i where $i = \pm 1, \pm 2, \ldots$; the use of positive and negative subscripts is as above.

The properties of the two infinite countable sequences, l_i and m_i, of the real and complex-conjugate eigenvalues as real α varies have been discussed by Lugt & Schwiderski (1965). In particular, note that $V(-p) = -V(p-2)$ and $W(-p) = -W(p-2)$ for all p so that $l_{-i} = 2-l_i$ and $m_{-i} = 2-m_i$. In fact $l_{-1} < 0$ if and only if $\alpha < 90°$ and $m_{-1} < 0$ if and only if $\alpha < 128.7°$.

Sternberg & Koiter (1958), Lugt & Schwiderski (1965) and Moffatt & Duffy (1980) have discussed various biharmonic problems in wedges relevant to the present problem. Their work shows that if conditions are imposed on ψ' at arcs $r = r_1$ and $r = r_2$ as well as the walls $\theta = \pm\alpha$ then ψ' may become large for fixed r as $r_1 \to 0$ or $r_2 \to \infty$ if $\alpha > 90°$. This may be described as the invalidity of St. Venant's principle in the context of elasticity or as spatial instability, in the sense that a small steady disturbance at $r = r_1$ may grow as r increases, or a disturbance at $r = r_2$ may grow as r decreases.

3.2. *The general case*

We anticipate the similar occurrence of families of even and odd eigenfunctions $\hat{\psi}$ for basic symmetric JH flows of types I, II_n and III_n with $R \neq 0$, families which reduce to (3.4a) and (3.5a) when $R = 0$, although the technical difficulties of solving the problem are greater for $R \neq 0$ than $R = 0$. Thus there is a set of even eigenfunctions with eigenvalues $\lambda = l_i$ and of odd eigenfunctions with eigenvalues $\lambda = m_i$ for each value of R, although in general $l_{-i} \neq 2-l_i$ and $m_{-i} \neq 2-m_i$ when $R \neq 0$. We conjecture that $l_1 > 2$ and $l_{-1} < 0$ if and only if $\alpha < \alpha_2(R)$ and that $m_1 > 2$ and $m_{-1} < 0$ if and only if $\alpha < \alpha_3(R)$, where α_2 and α_3 give the boundaries $\mathscr{B}_2$ and $\mathscr{B}_3$ of Fraenkel (1962). Indeed, we identify $\alpha_2(0) = 90°$ and $\alpha_3(0) = 128.7°$ in the biharmonic problem, and shall provide numerical and analytical results which are circumstantial evidence to support this conjecture. Further, following Moffatt & Duffy (1980, p. 311), we suggest that there is spatial instability of steady modes as $r \to \infty$ if $\lambda > 0$. This implies that the JH flows of types I, II_1 and III_1 are spatially stable but those of II_2, III_2, $\mathrm{II}_3, \ldots$ are unstable.

There are two complete sets of eigenfunctions ψ' so that the perturbation may be independently specified for $-\alpha \leqslant \theta \leqslant \alpha$ at both $r = r_1$ and $r = r_2$, subject to conservation of mass. It seems that the set belonging to the eigenvalues l_{-i} and m_{-i} is needed to represent the spatial growth of disturbances as r increases from r_1 and the set belonging to l_i and m_i to represent the growth as r decreases from $r = r_2$. It so happens that the θ-dependence of the two sets of eigenfunctions $r^\lambda\hat{\psi}(y)$ is the same when $R = 0$, but different when $R > 0$.

3.3. *The case* $\alpha \approx \alpha_2$

When $\alpha = \alpha_2(R)$ two solutions of the eigenvalue problem (3.3) may be seen by inspection:

$$\hat{\psi} = G_{0y}, \quad \lambda = 0 \tag{3.6a}$$

and

$$\hat{\psi} = G_{0y}, \quad \lambda = 2. \tag{3.6b}$$

These eigenvalues, valid for all R, can be identified with the first eigenvalues l_{-1} and l_1 respectively for the special case $R=0$. These solutions may be perturbed for small $\epsilon=\alpha-\alpha_2$ by first replacing y by y_2 in problem (3.3) so that it becomes

$$\hat{\psi}^{\mathrm{iv}}+\alpha_2^2\{\lambda^2+(\lambda-2)^2\}\hat{\psi}''+\alpha_2^4\lambda^2(\lambda-2)^2\hat{\psi} \\ -\alpha_2 R(\lambda-2)G'(\hat{\psi}''+\alpha_2^2\lambda^2\hat{\psi})+\alpha_2 R\lambda G'''\hat{\psi}+2\alpha_2 RG''\hat{\psi}'=0 \quad (3.7a)$$

and
$$\hat{\psi}=0,\quad \hat{\psi}'=0\quad \text{at } y_2=\pm\left(1+\frac{\epsilon}{\alpha_2}\right) \quad (3.7b)$$

without approximation, where a prime denotes differentiation with respect to y_2.

Now we have found that, for JH flows of types II_1 and II_2,

$$G=G_0+\epsilon^2G_2+\ldots \quad \text{as } \epsilon\to 0 \quad (3.8a)$$

for fixed R, where G_2 is specified by problem (2.18), so we expand

$$\lambda=\lambda_0+\epsilon\lambda_1+\epsilon^2\lambda_2+\ldots \quad (3.8b)$$

and
$$\hat{\psi}(y_2,\epsilon,R)=\psi_0(y_2,R)+\epsilon\psi_1(y_2,R)+\epsilon^2\psi_2(y_2,R)+\ldots \quad \text{as } \epsilon\to 0, \quad (3.8c)$$

and proceed to equate coefficients of successive powers of ϵ in (3.7).

The coefficients of ϵ^0 give

$$\psi_0^{\mathrm{iv}}+\alpha_2^2\{\lambda_0^2+(\lambda_0-2)^2\}\psi_0''+\alpha_2^4\lambda_0^2(\lambda_0-2)^2\psi_0 \\ -\alpha_2 R(\lambda_0-2)G_0'(\psi_0''+\alpha_2^2\lambda_0^2\psi_0)+\alpha_2 R\lambda_0 G_0'''\psi_0+2\alpha_2 RG_0''\psi_0'=0 \quad (3.9a)$$

and
$$\mathbf{B}\psi_0=\mathbf{0}. \quad (3.9b)$$

Two solutions of this problem are given in (3.6). We take $\psi_0=G_0'$ and $\lambda_0=0$, $\lambda_0=2$ in turn.

Case (*a*) $\lambda_0=0$

This solution joins up with the most rapidly growing solution when $R=0$. For this value of λ_0 we adopt the notation $\lambda_1=\lambda_1^{(0)}$.

The coefficients of ϵ give

$$\mathrm{L}\psi_1=4\lambda_1^{(0)}\alpha_2^2G_0''' \quad (3.10a)$$

and
$$\mathbf{B}\psi_1=\alpha_2^{-1}[0,0,G_0'''(-1),\,-G_0'''(1)]^{\mathrm{T}}. \quad (3.10b)$$

The solvability condition (2.13) for (3.10) leads to

$$\lambda_1^{(0)}=\frac{\{G_0'''(1)\}^2}{2\alpha_2^3\int_{-1}^{1}(G_0'')^2\,\mathrm{d}y_2}. \quad (3.11)$$

Note that $\lambda_1^{(0)}>0$, so that λ increases through zero as α increases through α_2, and the symmetric JH flows are spatially unstable for $\alpha>\alpha_2$ and stable for $\alpha<\alpha_2$. (Here we follow Fraenkel's 1962, p. 122 convention that $\alpha\geqslant 0$.)

We have already seen that when $R=0$, $\alpha_2=\frac{1}{2}\pi$ and $G_0=y+\pi^{-1}\sin\pi y$; then, (3.11) leads to $\lambda_1^{(0)}=4\pi^{-1}$. For the general case, however, Fraenkel (1962) showed that $\mathscr{B}_2$ may be expressed parametrically by

$$\alpha_2=(1-2m)^{\frac{1}{2}}K(m),\quad R=6(1-2m)^{-\frac{1}{2}}\{E(m)-(1-m)K(m)\}$$

for $0 \leqslant m < \frac{1}{2}$, where K is the complete elliptic integral of the first kind and E of the second kind with parameter m; and that

$$G_0'(y_2) = \frac{6mK^2(m)}{\alpha_2 R}\{1-\mathrm{sn}^2(K(m)\,y_2\,|\,m)\},$$

where sn is the Jacobian elliptic function. It follows at length from (3.11) that

$$\lambda_1^{(0)} = \frac{15m^2(1-m)^2K^3(m)}{4\alpha_2^3[m(1+m)\,K(m)-2(1-m+m^2)\,\{K(m)-E(m)\}]}.$$

We may recover the result for $R = 0$ from this by letting $m \to 0$; in addition we obtain

$$\lambda_1^{(0)} = c_1 R^3 + c_2 R + O(R^{-1}) \quad \text{as } R \to \infty,$$

where $c_1 = 5\{E(\frac{1}{2})-\frac{1}{2}K(\frac{1}{2})\}^{-4}/6912 = 0.02247$ and $c_2 = 42K(\frac{1}{2})\{E(\frac{1}{2})-\frac{1}{2}K(\frac{1}{2})\} = 0.7411$.

We have numerically integrated (2.5) for G_0 on $\mathscr{B}_2$ for $R = 20$ and $R = 30$ and find the values 194.65 and 628.88 respectively for $\lambda_1^{(0)}$ from (3.11). If we assume $\lambda_1^{(0)} = d_1 R^3 + d_2 R$ then these two values of $\lambda_1^{(0)}$ at $R = 20$ and 30 give rise to $d_1 = 0.02246$ and $d_2 = 0.7486$. We have also checked the form of (3.11) by numerically solving (3.3) to find λ for two pairs of values of α, R close to $\mathscr{B}_2$ and evaluating $\lambda_1^{(0)}$ from an assumed quadratic form of λ in ϵ.

For solutions G of types IV_1 and V_1 the calculation of λ_1 is a little different because $G_{\frac{1}{2}} \neq 0$. However, we show by a simple argument in the next section that, with the notation $\lambda_1 = \Lambda_1^{(0)}$, $\Lambda_1^{(0)} = -2\lambda_1^{(0)}$, and therefore

$$\Lambda_1^{(0)} = -\frac{\{G_0'''(1)\}^2}{\alpha_2^3\displaystyle\int_{-1}^{1}(G_0'')^2\,\mathrm{d}y_2} \tag{3.12}$$

for these asymmetric JH flows near the pitchfork bifurcation $\mathscr{B}_2$. Therefore they are spatially unstable for $\epsilon < 0$. The result (3.12) has also been checked numerically in the same way as was (3.11).

Case (*b*) $\lambda_0 = 2$

We next consider the perturbation about the eigensolution corresponding to $\lambda_0 = 2$. From equation (3.9*a*) we obtain

$$\mathrm{L}_2\psi_0 = 0, \tag{3.13a}$$

where we define the linear differential operator L_2 as

$$\mathrm{L}_2 = \frac{\mathrm{d}^4}{\mathrm{d}y_2^4} + 4\alpha_2^2\frac{\mathrm{d}^2}{\mathrm{d}y_2^2} + 2\alpha_2 R\left(G_0''\frac{\mathrm{d}}{\mathrm{d}y_2} + G_0'''\right).$$

Also, (3.9*b*) is

$$\mathbf{B}\psi_0 = \mathbf{0}. \tag{3.13b}$$

The solution of this problem is $\psi_0 = G_0'$, as anticipated.

The coefficients of ϵ in equation (3.7*a*) with the notation $\lambda_1 = \lambda_1^{(2)}$ give

$$\mathrm{L}_2\psi_1 = -4\alpha_2^2\lambda_1^{(2)}\{G_0''' - \alpha_2 R(G_0')^2\} \tag{3.14a}$$

and in (3.7*b*) give

$$\mathbf{B}\psi_1 = \alpha_2^{-1}[0, 0, G_0'''(-1), -G_0'''(1)]^{\mathrm{T}}. \tag{3.14b}$$

The generalized Lagrange identity for the operator L_2 is

$$\int_{-1}^{1} v\mathrm{L}_2 u\,\mathrm{d}y_2 = \int_{-1}^{1} u\mathrm{L}_2^{\dagger} v\,\mathrm{d}y_2 + [I_2(u,v)]_{-1}^{1}, \tag{3.15}$$

where
$$\mathrm{L}_2^{\dagger} = \frac{\mathrm{d}^4}{\mathrm{d}y_2^4} + 4\alpha_2^2\frac{\mathrm{d}^2}{\mathrm{d}y_2^2} - 2\alpha_2 RG_0''\frac{\mathrm{d}}{\mathrm{d}y_2} \tag{3.16}$$

and
$$I_2(u,v) = u'''v - u''v' + u'v'' - uv''' + 4\alpha_2^2(u'v - uv') + 2\alpha_2 RG_0'' uv. \tag{3.17}$$

Because the operator L_2 is not self-adjoint we need to use the solution $\psi^{\dagger}$ of the adjoint problem

$$\mathrm{L}_2^{\dagger}\psi^{\dagger} = 0 \tag{3.18a}$$

and
$$\mathbf{B}\psi^{\dagger} = \mathbf{0}; \tag{3.18b}$$

this solution is not known explicitly, but we note that $\psi^{\dagger}$, like ψ_0, is an even function.

The solvability condition for (3.14) gives

$$\lambda_1^{(2)} = \frac{G_0'''(1)\,\psi^{\dagger''}(1)}{2\alpha_2^3\displaystyle\int_{-1}^{1}\{G_0''' - \alpha_2 R(G_0')^2\}\,\psi^{\dagger}\mathrm{d}y_2} \tag{3.19}$$

because G_0 is an odd function and $\psi^{\dagger}$ an even function.

We have integrated the system for G_0 and $\psi^{\dagger}$ numerically for $R = 20$, $\alpha = \alpha_2$ and find that (3.19) leads to $\lambda_1^{(2)} = -144.83$. This has been checked in the same way as the analogous result when $\lambda_0 = 0$.

We also note that if $R = 0$ then $\alpha_2 = \frac{1}{2}\pi$ and $\psi^{\dagger} = 1 + \cos \pi y$. This in turn leads to $\lambda_1^{(2)} = -4\pi^{-1}$.

In addition to the numerical results referred to above we found some sample eigenvalues by numerically integrating the eigenproblem specified by systems (2.2) and (3.3). As in the previous numerical work we used the 'shooting' method (with normalization $\hat{\psi}''(-1) = 1$) to obtain solutions and, to start the tabulation of $\lambda(\alpha)$ for a given R, we related the calculation to the eigensolution at $R = 0$ to ensure confidence in identifying the appropriate eigenvalue. As will be appreciated, because it is a two-parameter (α, R) problem, a complete tabulation is not feasible. It will be convenient to recall the eigenvalue investigations of Lugt & Schwiderski (1965) and of Moffatt & Duffy (1980) for the special case $R = 0$ where the eigenvalues are determined from trigonometric equations, and we have encapsulated their results in figure 2. Because $V(-p) = -V(p-2)$ and $W(-p) = -W(p-2)$ (see (3.4) and (3.5)) there is symmetry about the line where the ordinate equals one.

It was decided to determine the effect on both l_1 and l_{-1} by changing to a small positive value of R, and we chose 0.3. To explain this choice of R, and indeed to help with the interpretation of the numerical results for $R > 0$, it is important to recall some properties of the JH flow. To this end we show, by way of an (R, α)-plane diagram in figure 3, the important boundaries which demarcate the various flow types. The latter figure is based on results from Fraenkel (1962) and Buitrago (1983). To appreciate the implications of figure 3 note that we have introduced a state variable, $\xi = 1 - \exp\{G'(0)\}^{-1}$, and give in figure 4 a perspective view of the principal solution sheets representing symmetric JH solutions for $\alpha > 0$, $R > 0$. Figure 5 contains sketches of sections of the sheets at $R = 1.6$, 0.3 and -0.5. We have labelled the sheets in figures 4 and 5 to make it easier to distinguish and visualize the

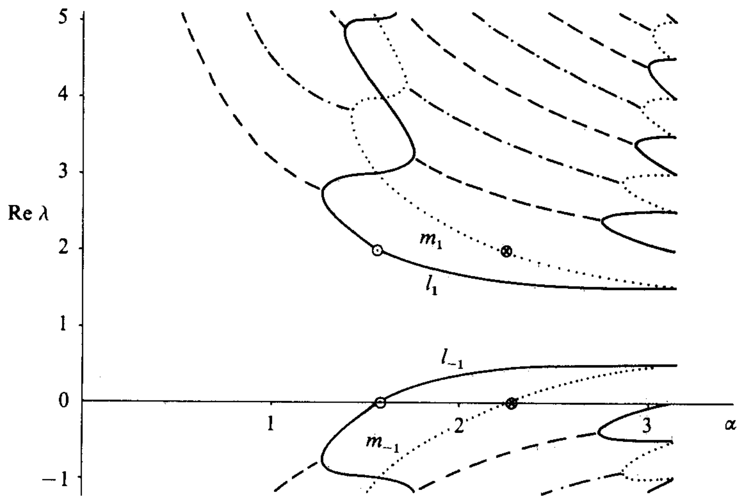

FIGURE 2. Eigenvalues for $R = 0$ from Lugt & Schwiderski (1965). ——, Real eigenvalues l_i of antisymmetric modes; ····, real eigenvalues m_i of symmetric modes; – – – –, real part of eigenvalues l_i of antisymmetric modes; –·–·–, real part of eigenvalues m_i of symmetric modes; $\odot$, point of zero skin-friction in Stokes solution (i.e. $\alpha_2(0)$); $\otimes$, singular point of Stokes solution (i.e. $\alpha_3(0)$).

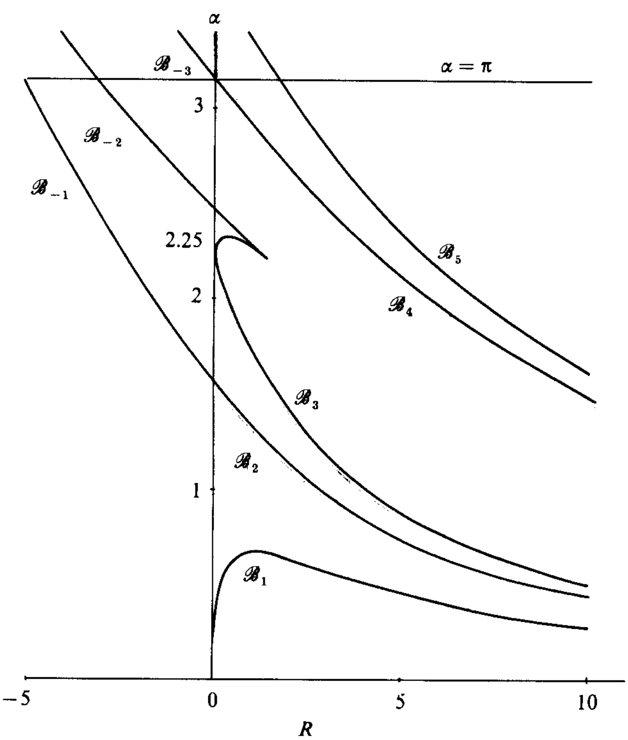

FIGURE 3. Boundaries corresponding to special properties of the Jeffery–Hamel solutions (after Fraenkel 1962 and Buitrago 1983). Note that $G''(-1) = 0$ on $\mathscr{B}_2$ and $\mathscr{B}_{-1}$; $(\partial G/\partial \alpha)^{-1} = 0$ on $\mathscr{B}_3$, $\mathscr{B}_{-2}$ and $\mathscr{B}_5$; and $G'(0) = G''(-1) = 0$ on $\mathscr{B}_4$ and $\mathscr{B}_{-3}$.

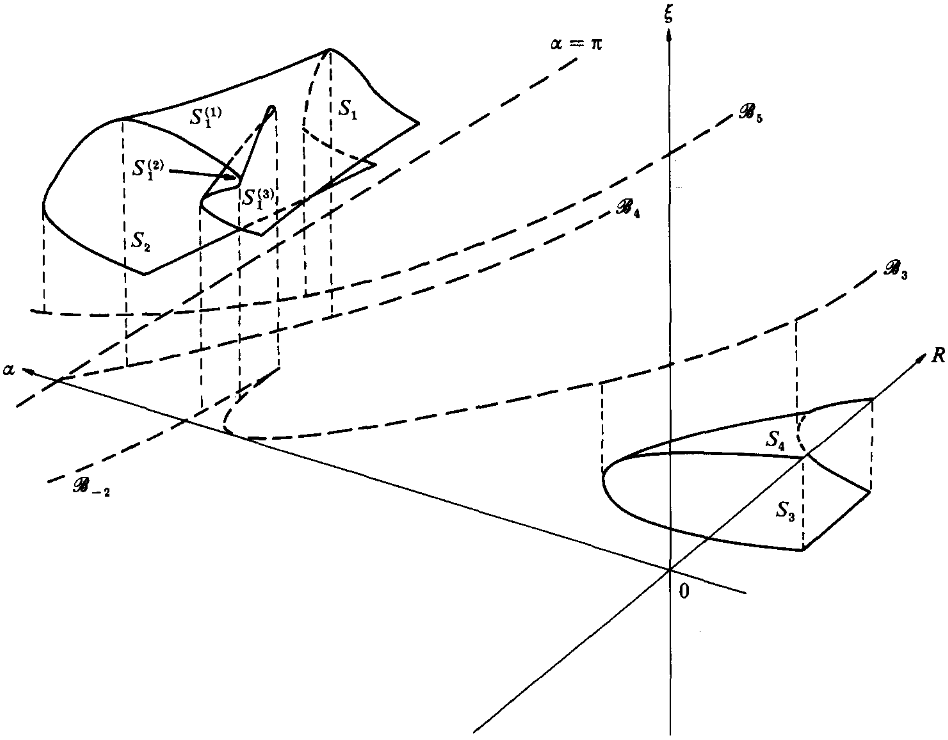

FIGURE 4. Perspective sketch of two parts of the solution sheets of symmetric Jeffery–Hamel flows: a region near the cusp and a region with α small. ($\xi = 1-\exp\{G'(0)\}^{-1}$.)

nature of the sheets in the neighbourhood of the cusp at $\alpha \approx 2.2$, $R \approx 1.4$. To understand this cusp it may help to look at figure 4 and think of the cusp that arises from the projection of the vertical tangent planes to a surface in catastrophe theory.

The eigenvalues, $l_{\pm 1}$, for $R = 0.3$ are given in figure 6 and the points representing the boundaries delineated in figure 3 are marked accordingly. We note that for this value of R, the two branches l_1 and l_{-1} which were separate when $R = 0$, now join near $\alpha = 2$ although for l_1 and l_{-1} lying between approximately $\frac{1}{2}$ and $\frac{3}{2}$ the JH flow is of type II_2. The tabulation of both l_1 and l_{-1} was continued from $\alpha = 2$ with decreasing α until the loci of l_1 and l_{-1} versus α develop vertical tangents and, presumably, merged into l_2 and l_{-2}, and also spawned complex eigenvalues – this is similar to the behaviour when $R = 0$. Even with this modest value of the Reynolds number the change in the eigenvalue pattern is relatively large. However, for the first eigenvalues emanating from the $\alpha = \pi$ region there is no joining of the l_1 and l_{-1} branches for $\alpha > 2.1$. We did not continue far with the tabulation of l_1 and l_{-1} after 'reflection' from the $\mathscr{B}_3$ boundary because the JH flow was developing large shears (of order 300).

The effect on l_1 and l_{-1} of further increasing R was also investigated. We chose a value $R = 1.6$ which is large enough to avoid the cusp in the boundary $\mathscr{B}_3$ (see figure 3). We note again the joining of the two branches l_1 and l_{-1} in the vicinity of $\alpha = 1.5$ and also the very large changes from the $R = 0$ pattern. The symmetry that exists for $R = 0$ is destroyed for $R > 0$. We have tabulated the l_{-1} branch of eigenvalues that emanates from the region of $\alpha = \pi$ and have displayed the results in figure 6 over the range $1 \leqslant \alpha \leqslant 3$. However, the calculation of the l_1 branch proved to be difficult and time-consuming and we obtained very little information.

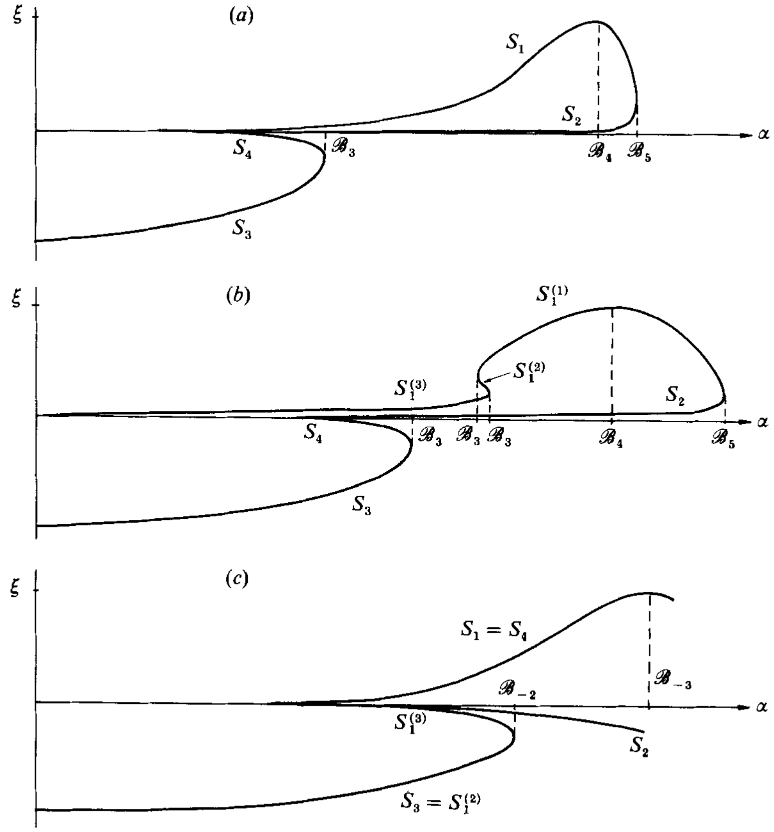

FIGURE 5. Sheet sections of the simpler Jeffery–Hamel solutions at (*a*) $R = 1.6$, (*b*) 0.3 and (*c*) -0.3. ($\xi = 1 - \exp\{G'(0)\}^{-1}$.) Note: these are sketches – certain aspects have been exaggerated.

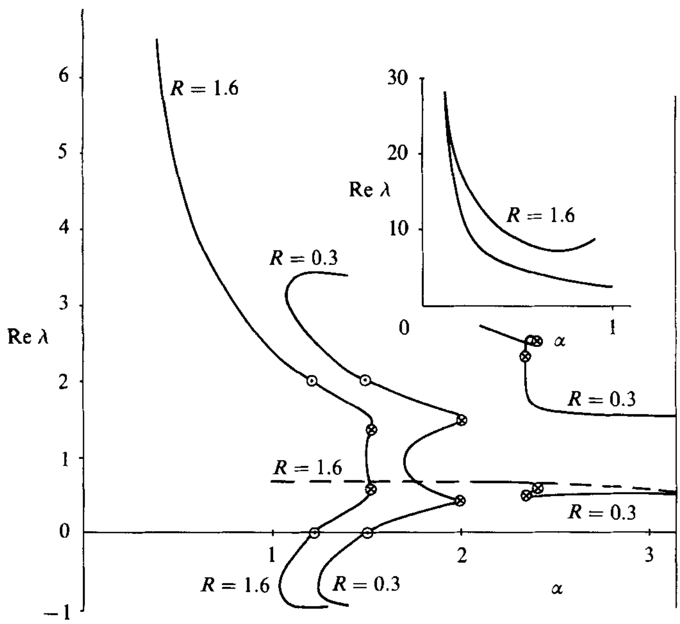

FIGURE 6. Real eigenvalues corresponding to the antisymmetric modes for $R = 0.3$ and 1.6 (cf. figure 2). The points marked ⊗ indicate $\alpha_3(R)$ and those marked ⊙ indicate $\alpha_2(R)$.

4. Steady flow in a channel: weakly nonlinear perturbation of Jeffery–Hamel flows

We described in §2 Jeffery–Hamel flows and their asymptotic behaviour near their pitchfork bifurcation. Our results may be summarized by the formulae.

$$\psi = \tfrac{1}{2}Q(G_0 + \delta\psi'), \tag{4.1}$$

$$\psi' \sim AG_0' \tag{4.2}$$

and
$$0 = \epsilon\delta A + \frac{\delta^3 A^3}{-a^2} + o(\epsilon\delta) \quad \text{as } \delta, \epsilon \to 0, \tag{4.3}$$

where we here define δ (not uniquely of course) to be of the magnitude of the perturbation of the JH stream function at the bifurcation $\mathscr{B}_2$ so that ψ' is of order one as $\delta \to 0$. Note that $a^2 < 0$ so that we interpret (4.3) as giving $\delta = |\epsilon|^{\frac{1}{2}}$ and $A = 0$ or $A = \pm A_e$ for $\epsilon < 0$, where $A_e = (-a^2)^{\frac{1}{2}}$. Thus the symmetric JH flows of types II_1 and II_2 come from the root $A = 0$ as $\epsilon \to 0$ and the asymmetric flows of types IV_1 and V_1 come from the roots $A = \pm A_e$ as $\epsilon \uparrow 0$.

We described in §3 the steady spatial linear modes of JH flows. Those of our results relevant to the perturbation of a JH flow near the pitchfork bifurcation may be summarized by (4.1) and (4.2), where here (4.3) is replaced by

$$\delta r \frac{\mathrm{d}A}{\mathrm{d}r} = \epsilon\lambda_1^{(0)}\,\delta A + o(\epsilon\delta A) \quad \text{as } \epsilon \to 0 \tag{4.4}$$

because we found that $A \propto r^\lambda$ and $\lambda \sim \epsilon\lambda_1^{(0)}$ as $\epsilon \to 0$, and by (4.1) with

$$\psi' \sim r^2 BG_0' \tag{4.5}$$

and
$$\delta r \frac{\mathrm{d}B}{\mathrm{d}r} = \epsilon\lambda_1^{(2)}\delta B + o(\epsilon\delta B) \quad \text{as } \epsilon \to 0 \tag{4.6}$$

because we found that $B \propto r^\lambda$ and $\lambda \sim 2 + \epsilon\lambda_1^{(2)}$ as $\epsilon \to 0$.

In this section we shall combine the effects of the weakly nonlinear perturbation of JH flows and of their linear spatial perturbation near the bifurcation at $\mathscr{B}_2$ and add the effects of small curvature of the walls, i.e. we shall synthesize and generalize the perturbation theories of §§2 and 3 to describe steady flows along a channel. At the simplest, a combination of the effects gives (4.1) and (4.2), where A satisfies the Landau-like equation

$$\frac{\mathrm{d}A}{\mathrm{d}\rho} = k_1 A + k_2 A^3, \tag{4.7}$$

and we define $\rho = \delta^2 \ln r, k_1 = \epsilon\lambda_1^{(0)}/\delta^2$ and $k_2 = \lambda_1^{(0)}/(-a^2)$. It can be seen that (4.7) reduces to (4.3) or (4.4) in the appropriate special case. We may also anticipate a similar Landau-like equation for B, but there may be interactions between the two modes, and indeed with other modes for which we have no convenient analytical solutions.

However, we may reach (4.7), and its analogue for B, more rationally. We seek the distinguished limit whereby δ, ϵ and the magnitude of the small wall-curvature are related so that the effects of weak nonlinearity, weak spatial growth and small curvature are of the same order of magnitude. The arguments above suggest that we use the method of multiple scales with the 'slow' radial coordinate ρ as well as the 'fast' one r. Thus we take the equation of the walls to be $\theta = \pm\alpha_2\{1 + \delta^2 h_\pm(\rho)\}$ for given functions $h_\pm$ and replace $r\partial/\partial r$ by $r\partial/\partial r + r(\partial\rho/\partial r)\,\partial/\partial\rho = r\partial/\partial r + \delta^2\partial/\partial\rho$ in (2.1). Then (2.1) and (4.1) lead to

$$\nabla^4\psi' + \frac{R}{\alpha_2^3 r^4}\{G_0'''(r\psi_r' + \delta^2\psi_\rho') + 2G_0''\,\psi_{y_2}'\} - \frac{RG_0'}{\alpha_2 r^2}\{r(\nabla^2\psi')_r + \delta^2(\nabla^2\psi')_\rho\}$$
$$= -\frac{\delta R}{\alpha_2 r^2}[(r\psi_r' + \delta^2\psi_\rho')(\nabla^2\psi')_{y_2} - \psi_{y_2}'\{r(\nabla^2\psi')_r + \delta^2(\nabla^2\psi')_\rho\}] \quad (4.8a)$$

and

$$\psi' = \frac{\pm 1 - G_0}{\delta}, \quad \psi_{y_2}' = \pm\frac{\delta^4 h_\pm'}{r}(r\psi_r' + \delta^2\psi_\rho') - \frac{G_0'}{\delta} \quad \text{at } y_2 = \pm\{1 + \delta^2 h_\pm(\rho)\}, \quad (4.8b)$$

where $\nabla^2 = \partial^2/\partial r^2 + \partial/r\,\partial r + \partial^2/r^2\,\partial\theta^2$ is the Laplacian expressed in terms of the fast variable r. We may expand

$$\psi' = \psi_0 + \delta\psi_1 + \delta^2\psi_2 + \dots \quad \text{as } \delta \to 0 \quad (4.9)$$

where

$$\psi_0(r, y_2) = A(\rho)\,G_0'(y_2) + B(\rho)\,r^2 G_0'(y_2). \quad (4.10)$$

This takes into account not only the self-interaction of each component but also their mutual interaction. We find $A, B, \psi_1, \psi_2, \dots$ by the methods of §§2 and 3. The results as well as the methods are similar, for we find at length that A satisifies a Landau equation of the form (4.7), where

$$k_1(\rho) = \frac{\{G_0'''(1)\}^2\{h_+(\rho) + h_-(\rho)\}}{4\alpha_2^2 \int_{-1}^{1} (G_0'')^2 \,\mathrm{d}y_2} \quad (4.11)$$

and

$$k_2 = \frac{R\int_{-1}^{1} F_1'(G_0'')^2 \,\mathrm{d}y_2}{2\alpha_2 \int_{-1}^{1} (G_0'')^2 \,\mathrm{d}y_2}. \quad (4.12)$$

If $h_\pm = \epsilon/(\alpha_2\delta^2)$ then we recover the results of §§2 and 3 with $k_1 = \epsilon\lambda_1^{(0)}/\delta^2$ in (4.7). We saw that the solution $A = \pm A_\mathrm{e}$, where $A_\mathrm{e} = (-k_1/k_2)^{\frac{1}{2}}$ for $\epsilon < 0$ gave the asymmetric JH flows of types IV_1 and V_1 as $\epsilon \uparrow 0$. Linearizing (4.7) about its solution $A = A_\mathrm{e}$ we find

$$\frac{\mathrm{d}(A - A_\mathrm{e})}{\mathrm{d}\rho} = -2k_1(A - A_\mathrm{e}),$$

so that $A - A_\mathrm{e} \propto \exp(-2\epsilon\lambda_1^{(0)}\rho/\delta^2) = r^{-2\epsilon\lambda_1^{(0)}}$. This shows that the spatial growth of normal modes of the asymmetric JH flows is of the opposite sign and twice the magnitude of the growth of modes of the symmetric JH flows, as anticipated in (3.12).

Equation (4.7) also describes the development of a disturbance in a channel with walls $\theta \approx \pm\alpha_2(R)$ provided that k_1 is chosen to represent the linear growth of the least stable spatial mode. Equation (4.7) has the general solution given by

$$A^{-2}(\rho) = -2k_2 \exp\left\{-2\int^{\rho} k_1(\rho')\,\mathrm{d}\rho'\right\}\int_0^{\rho} \exp\left\{2\int^{\rho'} k_1(\rho'')\,\mathrm{d}\rho''\right\}\mathrm{d}\rho',$$

but it is perhaps more helpful to use its qualitative behaviour to indicate how A increases or decreases. Suppose, for example, we take a slowly diverging channel such that $k_1(\rho) < 0$ upstream of the critical station ρ_0, where $h_+(\rho_0) + h_-(\rho_0) = 0$, $k_1(\rho_0) = 0$ and $k_1(\rho) > 0$ for $\rho > \rho_0$, in order to approximate the channels used for the numerical solutions of the Navier–Stokes equations by Cliffe & Greenfield (1982) and Sobey & Drazin (1986). Then (4.7) shows that if the disturbance, and thence A, is specified upstream of the critical station then A decreases for $\rho < \rho_0$ but increases for $\rho > \rho_0$ superexponentially until it becomes infinite. (Of course, (4.7) would become invalid before A became infinite.) There is a paradox here, as first suggested by Sobey & Drazin, because the numerical solutions of the Navier–Stokes equations give equilibration of disturbances downstream of the critical station as if the constant k_2 in (4.12) were negative. However, it seems that the flow downstream of the critical station as calculated numerically is not close to a local JH solution, so perhaps A begins to grow for $\rho > \rho_0$ in accord with (4.7) but soon becomes large, so that (4.7) becomes invalid, and thereafter the flow approaches a new equilibrium not approximated locally by any JH flow.

We find also that B satisfies

$$\frac{\mathrm{d}B}{\mathrm{d}\rho} = k_3 B + k_4 A^2 B, \tag{4.13}$$

where

$$k_3(\rho) = \frac{-G_0'''(1)\,\psi^{\dagger''}(1)\,\{h_+(\rho)+h_-(\rho)\}}{4\alpha_2^2 \int_{-1}^{1} \{\alpha_2 R(G_0')^2 - G_0'''\}\,\psi^\dagger\,\mathrm{d}y_2}, \tag{4.14}$$

$$k_4 = \frac{R\int_{-1}^{1} (G_0'' F_2 + G_0' F_1'')'\psi^\dagger\,\mathrm{d}y_2}{2\alpha_2 \int_{-1}^{1} \{\alpha_2 R(G_0')^2 - G_0'''\}\,\psi^\dagger \mathrm{d}y_2}, \tag{4.15}$$

F_1 is defined by (2.15) and F_2 satisfies

$$\mathrm{L}_2 F_2 = -2\alpha_2 R(G_0' G_0''')', \tag{4.16a}$$

$$\mathbf{B}F_2 = \mathbf{0}. \tag{4.16b}$$

If $h_\pm = \epsilon/(\alpha_2\delta^2)$ we recover the results of §3 with $k_3 = \epsilon\lambda_1^{(2)}/\delta^2$ in (4.13).

5. The temporal stability of Jeffery–Hamel flow

We shall next examine the linear stability of a JH flow to two-dimensional disturbances. We write $\psi = \frac{1}{2}Q(G+\delta\Psi')$ and linearize (2.1) for small perturbations. Therefore

$$\nabla^2\Psi_t' + \frac{RG_y}{\alpha r}(\nabla^2\Psi')_r - \frac{RG_{yyy}}{\alpha^3 r^3}\Psi_r' - \frac{2RG_{yy}}{\alpha^2 r^4}\Psi_\theta' = \nabla^4\Psi' \tag{5.1a}$$

and

$$\Psi' = 0, \quad \Psi_\theta' = 0 \quad \text{at } \theta = \pm\alpha \tag{5.1b}$$

after scaling the time variable. Using the method of normal modes, take $\Psi'(r,\theta,t,\alpha,R) = \mathrm{e}^{st}\,\hat{\Psi}(r,\theta,\alpha,R)$. Therefore

$$\nabla^4\hat{\Psi} - \frac{RG_y}{\alpha r}(\nabla^2\hat{\Psi})_r + \frac{RG_{yyy}}{\alpha^3 r^3}\hat{\Psi}_r + \frac{2RG_{yy}}{\alpha^2 r^4}\hat{\Psi}_\theta = s\nabla^2\hat{\Psi} \tag{5.2a}$$

and

$$\hat{\Psi} = 0, \quad \hat{\Psi}_\theta = 0 \quad \text{at } \theta = \pm\alpha. \tag{5.2b}$$

5.1. *The case* $R = 0$

It is instructive to put $R = 0$ in (5.2*a*) so that

$$\nabla^4 \hat{\Psi} = s\nabla^2 \hat{\Psi}. \tag{5.3a}$$

It follows that

$$s \iint_D |\nabla \hat{\Psi}|^2 r \,\mathrm{d}r\,\mathrm{d}\theta = -s \iint_D \hat{\Psi}^* \nabla^2 \hat{\Psi} r \,\mathrm{d}r\,\mathrm{d}\theta$$
$$= -\iint_D \hat{\Psi}^* \nabla^4 \hat{\Psi} r \,\mathrm{d}r\,\mathrm{d}\theta$$
$$= -\iint_D |\nabla^2 \hat{\Psi}|^2 r \,\mathrm{d}r\,\mathrm{d}\theta,$$

on use of Gauss's divergence theorem, provided that there is no contribution from the 'boundaries' at $r = 0, \infty$. Here D is the domain of flow $\{(r,\theta)\,|\,0 \leqslant r < \infty, -\alpha \leqslant \theta \leqslant \alpha\}$ and $\hat{\Psi}^*$ is the complex conjugate of $\hat{\Psi}$. Therefore $s < 0$ and we may rescale $r_* = r(-s)^{\frac{1}{2}}$ so that (5.3*a*) becomes

$$\nabla_*^4 \hat{\Psi} = -\nabla_*^2 \hat{\Psi}, \tag{5.3b}$$

where $\nabla_*^2 = \partial^2/\partial r_*^2 + \partial/r_* \partial r_* + \partial^2/r_*^2 \partial\theta^2$. From the system (5.3*b*) and (5.2*b*) we note that either all negative s are eigenvalues or there is no eigenvalue. In short, it seems that the Stokes flow is stable, although the eigenvalue problem is degenerate.

However, the boundary conditions at $r = 0, \infty$ deserve closer attention in the light of the above paragraph and of our discussion of the case $R = 0$ in §3. To be specific, *add* the boundary conditions

$$\hat{\Psi} = 0, \quad \hat{\Psi}_r = 0 \quad \text{at } r = r_1, r_2 \tag{5.3c}$$

to (5.2*b*), to pose the eigenvalue problem (5.3*a*), (5.2*b*) and (5.3*c*) to determine s. Dimensional analysis leads to

$$s = r_2^{-2} P\left(\frac{r_1}{r_2}\right)$$

for some function P, which may arise as a sequence of eigenvalues. We may define a new radial coordinate r/r_2 or equivalently just take $r_2 = 1$. Then the problem is

$$\nabla^4 \hat{\Psi} = P \nabla^2 \hat{\Psi} \quad \text{in } D_* \tag{5.4a}$$

and

$$\hat{\Psi} = 0, \quad \frac{\partial \hat{\Psi}}{\partial n} = 0 \quad \text{on } \partial D_* \tag{5.4b}$$

where $D_* = \{(r,\theta)\,|\,r_1 \leqslant r \leqslant 1, \; -\alpha \leqslant \theta \leqslant \alpha\}$ is the domain of flow and ∂D_* its boundary. The usual variational formulation gives

$$P = -\min_{\psi \in H} \left(\frac{I_1}{I_2}\right), \tag{5.5}$$

where H is the space of well-behaved functions satisfying (5.4*b*) and

$$I_1 = \iint_{D_*} (\nabla^2 \psi)^2 r \,\mathrm{d}r\,\mathrm{d}\theta, \quad I_2 = \iint_{D_*} (\nabla \psi)^2 r \,\mathrm{d}r\,\mathrm{d}\theta.$$

We may use the Rayleigh–Ritz method to calculate P or simply estimate the first mode with a single trial function. To the latter end we take a separable form

$\alpha \backslash r_1$	0.1	0.5	0.9
0.5	−122.366	−99.6775	−3877.28
1.0	−34.0908	−134.632	−3930.79
2.0	−40.8323	−152.581	−3944.40
3.0	−46.5893	−156.483	−3946.93

TABLE 1. Upper bounds as estimates of P by use of the trial functions $f(r) = 1+\cos\{\pi(2r-1-r_1)/(1-r_1)\}$ and $g(\theta) = 1+\cos(\pi\theta/\alpha)$ in (5.5)–(5.7) for various values of α and r_1

$\psi = f(r)g(\theta)$ for such an estimate, where $f(r_1) = f'(r_1) = f(1) = f'(1) = 0$ and $g(\alpha) = g'(\alpha) = g(-\alpha) = g'(-\alpha) = 0$. (We note that the separable form is appropriate in the limits as $\alpha \to 0$, $r_1 \to 1$ – we return to this point briefly in §6.) With this form for ψ we find

$$I_1 = \int_{r_1}^1 \left(f''+\frac{f'}{r}\right)^2 r\,\mathrm{d}r \int_{-\alpha}^{\alpha} g^2\,\mathrm{d}\theta + 2\int_{r_1}^1 \frac{f}{r}\left(f''+\frac{f'}{r}\right)\mathrm{d}r \int_{-\alpha}^{\alpha} gg''\,\mathrm{d}\theta + \int_{r_1}^1 \frac{f^2}{r^3}\,\mathrm{d}r \int_{-\alpha}^{\alpha} g''^2\,\mathrm{d}\theta, \tag{5.6}$$

$$I_2 = \int_{r_1}^1 f'^2 r\,\mathrm{d}r \int_{-\alpha}^{\alpha} g^2\,\mathrm{d}\theta + \int_{r_1}^1 \frac{f^2}{r}\,\mathrm{d}r \int_{-\alpha}^{\alpha} g'^2\,\mathrm{d}\theta. \tag{5.7}$$

In the special case when α is small, further progress is possible; we find that

$$P \sim -\min\left\{\frac{\displaystyle\int_{r_1}^1 \frac{f^2}{r^3}\,\mathrm{d}r \int_0^1 \left(\frac{\mathrm{d}^2 g}{\mathrm{d}y^2}\right)^2 \mathrm{d}y}{\displaystyle\alpha^2 \int_{r_1}^1 \frac{f^2}{r}\,\mathrm{d}r \int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}y}\right)^2 \mathrm{d}y}\right\} \quad \text{as } \alpha \to 0,$$

where $\theta = \alpha y$. Consider now the further limit $r_1 \to 1$. We find that

$$P \sim -\min\left\{\frac{\displaystyle\int_0^1 \left(\frac{\mathrm{d}^2 g}{\mathrm{d}y^2}\right)^2 \mathrm{d}y}{\displaystyle\alpha^2 \int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}y}\right)^2 \mathrm{d}y}\right\} \quad \text{as } \alpha \to 0, r_1 \to 1$$

in that order. It can be shown by the calculus of variations that the required extremal is given by $g(\theta) = 1+\cos \pi y$ so that

$$P \sim -\frac{\pi^2}{\alpha^2} \quad \text{as } \alpha \to 0, \text{and } r_1 \to 1.$$

Numerical evaluations of the integrals in (5.6) and (5.7) using the trial functions $f(r) = 1+\cos\{\pi(2r-1-r_1)/(1-r_1)\}$ and $g(\theta) = 1+\cos(\pi\theta/\alpha)$ gives the values for P shown in table 1. With $\alpha = 10^{-4}$, $r_1 = 0.99$ we find by numerical integration that $\alpha^2 P = -9.96$. We have also repeated the analysis and numerical evaluation of I_1 and I_2 with $f(r) = (r-1)^2(r-r_1)^2$ and $g(\theta) = (\alpha^2-\theta^2)^2$, the results of which show the same general trends.

5.2. *The general case*

The case above not only is of some intrinsic interest but also indicates how to use the boundary conditions when $R > 0$. So, although there is always stability when $R = 0$, the solution of this case sheds a little light on the nature of the instability and on how the instability depends on conditions near and far from the line source when $R > 0$. Indeed, if $s \neq 0$ then we may rescale $r_* = r|s|^{\frac{1}{2}}$ in (5.2) and deduce that either all magnitudes of s give eigenvalues or none do. If $s = 0$ then we may revert to the treatment of small steady perturbations of a Jeffery–Hamel flow.

5.3. *The case $\alpha \approx \alpha_2$*

Equations (3.6) give two solutions of problem (5.2), namely

$$\hat{\Psi} = G_{0y}, \quad s = 0, \quad \alpha = \alpha_2(R) \tag{5.8a}$$

and

$$\hat{\Psi} = r^2 G_{0y}, \quad s = 0, \quad \alpha = \alpha_2(R), \tag{5.8b}$$

for all R. Thus $s = 0$ is at least a double eigenvalue. For the reasons outlined in the previous paragraph, it appears fruitless to perturb this eigenvalue without specifying the boundary conditions for small and large r.

6. Conclusions

The linear and weakly nonlinear theories of the preceding sections suggest various physical results. Fraenkel (1962, p. 133, 1963, 1973) recognized and proved that the approximation to steady flow along a channel with walls of small curvature by symmetric Jeffery–Hamel flows locally is valid where $\alpha(x) < \alpha_3(R)$. We have shown, however, that his assumption of symmetric flow is invalid in practice, because small asymmetric steady disturbances of a symmetric Jeffery–Hamel flow grow spatially unless $\alpha < \alpha_2(R)$. So re-application of Fraenkel's arguments implies that the local approximation is valid only if $\alpha(x) < \alpha_2(R)$ everywhere. We have also gone a little further, showing heuristically in §4 that if $\alpha(x)$ does not exceed $\alpha_2(R)$ either by too much or for too far downstream then weakly nonlinear asymmetric disturbances may grow but will ultimately decay downstream where $\alpha(x) < \alpha_2(R)$.

This conclusion was based on our interpretation of the linear spatial stability in §3. Taking a steady perturbation ψ' of the stream function of a basic Jeffery–Hamel flow as a mode such that $\psi' \propto r^{\lambda}$, we deemed the eigenvalues $\lambda > 0$ as spatially unstable as $r \to \infty$, the perturbation growing downstream and dominating the basic flow there. In addition we interpreted the eigenvalues that decreased below 2 as unstable as $r \to 0$; the intuitive justification of this is less clear, although it may be noted that the pressure perturbation $p' \propto r^{\lambda-2}$, so that $p' \to \infty$ as $r \to 0$ if $\lambda < 2$. However, these arguments are no more than plausible, and the strongest support for our interpretation comes from the extensive analysis of the case $R = 0$ in the literature.

Sternberg & Koiter (1958) posed and solved a boundary-value problem modelling the plane elastic deformation of a wedge, which happens to be the complete analogy of a problem of Stokes flow. They used a Mellin transform, obtaining the eigenvalue problem (3.3) for the special case $R = 0$, and solved the boundary-value problem explicitly. They interpreted their results as the breakdown of St. Venant's principle when $\alpha > \frac{1}{2}\pi$, because then a small local perturbation of the solution near $r = 0$ influences, and is influenced by, the solution for large r. Moffatt & Duffy (1980) solved a similar boundary-value problem to model Stokes flow in a wedge and indicated the analogous interpretation of perturbations of Jeffery–Hamel flows, although like Fraenkel (1962, 1963) they confined their attention to symmetric perturbations. The result that flow far upstream interacts with flow far downstream in a channel if $\alpha > \frac{1}{2}\pi$ is, of course, intuitive, because a Jeffery–Hamel flow is clearly not a locally valid approximation if a wall bends backwards where its tangent is parallel to the z-axis (see figure 1).

We have recognized that $\frac{1}{2}\pi = \alpha_2(0)$ and that $\lambda = 0, 2$ are eigenvalues of the problem (3.3) if $\alpha = \alpha_2(R)$ for all R. So we have generalized the interpretation of the significance of the values 0, 2 of λ and of the breakdown of the local approximation where $\alpha(x) > \alpha_2(R)$, although we have not solved any specific boundary-value problem for $R > 0$ because the Mellin transformation is no longer helpful. This is the essence of our heuristic argument to conjecture the significance of the angle $\alpha_2(R)$.

The following problem may help to focus the issue. Suppose that ψ satisfies the vorticity equation (2.1*a*) and boundary conditions

$$\psi = \pm\tfrac{1}{2}Q, \quad \psi_\theta = 0 \quad \text{at } \theta = \pm\alpha \quad \text{for } r_1 < r < r_2$$

and

$$\psi = \tfrac{1}{2}Q\left\{-1 + \int_{-\alpha}^{\theta} U_i(\theta)\,\mathrm{d}\theta\right\} \quad \text{at } r = r_i \quad \text{for } -\alpha < \theta < \alpha,$$

where U_1 and U_2 are given function such that

$$\int_{-\alpha}^{\alpha} U_i(\theta)\,\mathrm{d}\theta = 2 \quad \text{for } i = 1, 2.$$

We conjecture that the solution in general approaches the appropriate Jeffery–Hamel solution for fixed $r > 0$ in the limit as both $r_1 \to 0$ and $r_2 \to \infty$ if $\alpha < \alpha_2(R)$ but not, in general, if $\alpha > \alpha_2(R)$. This conjecture is in part based on the above arguments and in part on Brady's (1984) numerical results for an analogous problem of steady flow which is driven by suction through the parallel porous walls of a two-dimensional channel. We hope to publish more about this point soon.

Then what is the nature and explanation of flow in a channel when $\alpha(x) > \alpha_2(R)$ for some substantial distance downstream? We have the analysis and discussion above together with laboratory experiments (Cherdron, Durst & Whitelaw 1978; Sobey 1985; Sobey & Drazin 1986, etc.) and numerical experiments (Cliffe & Greenfield 1982 and Sobey & Drazin 1986) to help us answer this question. It is clear that the flow becomes asymmetric and there are regions of closed streamlines. The occurrence of closed streamlines may precede or succeed symmetry breaking of steady flow as R increases, but both arise at approximately the same value of R if the walls have small curvature. This, then, is an important case study of breakaway, which may occur at any value of R according to the shape of the channel. There is no local symmetric Jeffery–Hamel flow to approach in regions where $\alpha > \alpha_2$, so it seems that there is a strong interaction of the flow in the whole length of the channel where $\alpha(x) > \alpha_2(R)$ and the flow in this length must be determined as a whole. If $\alpha(x) < \alpha_2(R)$ far downstream then the flow will revert to the local symmetric Jeffery–Hamel flow of type I or II_1 there. Of course, if R is large enough for a given channel with distribution $\alpha(x)$ then time-periodic, quasi-periodic, chaotic, or turbulent flow may arise.

The analogy of the theory of stability of plane parallel flows and the present theory of the stability of Jeffery–Hamel flows should be noted. Problem (3.3) is the analogue of the Orr–Sommerfield problem for steady spatially growing modes. However, the analogy does not seem to open up a vista of results analogous to all the results of the

Orr–Sommerfeld and Rayleigh stability problems, because the technical difficulties associated with polar coordinates are much more severe than those with Cartesian coordinates; in particular, we have not been able to find modes by separating the variables of time or the Cartesian coordinate perpendicular to the plane of flow, let alone find an analogue of Squire's theorem to show that two-dimensional disturbances are the most unstable. Nevertheless, it seems promising to use (3.3*a*) with velocity profiles other than those of Jeffery–Hamel flows in order to find approximately the spatial development of unsteady modes, just as the Orr–Sommerfeld equation is used with profiles of flows that are not exact solutions of the Navier–Stokes equations. Indeed, the spatial stability of diverging flows such as two-dimensional jets may be profitably studied by using both the Orr–Sommerfeld equation and (3.3*a*); the former is suitable for jets that are nearly parallel and the latter for jets that are nearly radial, whereas real jets are intermediate between parallel and radial ones. Also, perturbations of the flow in a diverging channel may be better approximated by choosing G so that the local radial velocity is $\frac{1}{2}QG_y/(\alpha r)$ in order to use (3.3) to estimate the local spatial decay of steady modes.

We next relate our results to those for parallel flows, considering flows in the domain D_* of §5. In that limit as $\alpha \rightarrow 0$ for fixed $r_1 = 0$ and $r_2 < \infty$, Jeffery–Hamel flows of types I and III$_1$ become locally parallel at each station, with velocity $u = \frac{3}{2}(1-y^2)/r$ between the boundaries at $z = \pm L$, where $L = \alpha r$, $z = \theta r$ and $y = \theta/\alpha = z/L$. If $\alpha \ll 1-r_1 \ll 1$ then the flow is uniformly parallel in the domain D_* because $L \approx \alpha$ is constant for $r_1 \leqslant r \leqslant 1$. Thus we anticipate that the stability characteristics of JH flows may be found in the limit as $\alpha \rightarrow 0$ by use of a local approximation of plane Poiseuille flow and in the limit as α and $(1-r_1)/\alpha \rightarrow 0$ by a global approximation of plane Poiseuille flow.

Now steady small perturbation of plane Poiseuille flow can be represented in terms of spatial modes proportional to e^{ikx} for complex wavenumbers k, where $\mathrm{Im}\, k$ gives the relative decay of disturbances as x increases. The wavenumbers may be found as eigenvalues of the Orr–Sommerfeld problem for *steady* modes (cf. Bramley & Dennis 1982). On identifying x with r in the limit as α and $(1-r_1)/\alpha \rightarrow 0$ we may understand, as follows, why the JH spatial modes give $\mathrm{Re}\, l_{\pm i}$, $\mathrm{Re}\, m_{\pm i} \rightarrow \pm\infty$ as $\alpha \rightarrow 0$ (see e.g. figure 2 for the case $R = 0$). The spatial modes of the JH problem are proportional to $r^{\lambda} = \mathrm{e}^{\lambda \ln r} \sim \mathrm{e}^{\lambda \ln x}$ as $\alpha \rightarrow 0$. Although no single mode of the JH flow is identified with a single mode of the parallel flow, each mode of the JH flow is identifiable with some superposition of the spatial modes of the parallel flow, and vice versa, as $\alpha \rightarrow 0$. Therefore $l_i, m_i \rightarrow \pm\infty$ as $\alpha \rightarrow 0$ and $i \rightarrow \pm\infty$ in order that e^{ikx} may be equal to a sum of modes proportional to $\exp(l_i \ln x)$ and $\exp(m_i \ln x)$.

Also the temporal modes of plane Poiseuille flow are proportional to $\mathrm{e}^{ik(x-ct)}$ for complex eigenvalues c found in terms of real wavenumbers k by solving the Orr–Sommerfeld problem. It can be shown that

$$c = \frac{\nu}{kL^2}\left\{\frac{c_n^{(0)}}{\mathrm{i}} + \frac{QkL}{\nu}c_n^{(1)} + O\left(\frac{Q^2k^2L^2}{\nu^2}\right)\right\} \quad \text{as } \frac{QkL}{\nu} \rightarrow 0,$$

where Q/L is the velocity scale and the dimensionless velocities $c_n^{(0)}$ and $c_n^{(1)}$ are real and may be found in simple terms of the solution of a transcendental equation, for $n = 0, 1, 2, \ldots$ (cf. Drazin & Reid 1981, pp. 159–160). In fact $c_n^{(0)}(k) \geqslant c_n^{(0)}(0) = \frac{1}{4}(n+2)^2\pi^2$, so that the least-damped mode has relative growth rate

$$s = -\mathrm{i}kc \sim -\frac{\nu\pi^2}{L^2}.$$

This value is in dimensional terms. It gives $s = -\pi^2/\alpha^2$ as in §5 if we scale s with the factor ν and take $r_1 \approx r_2 = 1$.

In summary, our results, especially those of §5, and those quoted suggest that the stability of the steady symmetric basic flow in a wedge is as follows. (*a*) If $\alpha - \alpha_2(R)$ is negative and not small then the basic flow is a JH flow, it is stable, and the conditions at $r = r_1, r_2$ are only of local significance. (*b*) If $\alpha \approx \alpha_2(R)$ then the basic flow is at least approximately a JH flow, and it may be stable or unstable according to the precise boundary conditions at $r = r_1, r_2$. (*c*) If $\alpha - \alpha_2(R)$ is positive and not small then the basic flow is not a JH flow, it is unstable, and the conditions at $r = r_1, r_2$ determine the flow throughout the wedge, whereby an asymmetric flow may develop.

Finally, it should be remembered that real flows may be unsteady or three-dimensional, whereas the results discussed in this paper are for two-dimensional flows which are mostly steady. The possibility of time-periodic flows and knowledge of the well-known subcritical Hopf bifurcation of plane Poiseuille flow at $R = 3848$ suggest that there may be a curve $\mathscr{B}_0$, on which $\alpha = \alpha_0(R)$, say, in the (α, R)-plane of figure 3 at which Hopf bifurcations occur. (Note that plane Poiseuille flow arises as a limiting case, $\alpha = 0$, of Jeffery–Hamel flows, and that with our definition of R based on volume flux the maximum of a parabolic velocity distribution is $\frac{3}{2}$.) We speculate that this curve rises leftwards from the point $R = 3848$ on the R-axis and crosses the curve $\mathscr{B}_2$; thus the first instability as R increases is at a Hopf bifurcation if the value of α is less than that at the intersection of $\mathscr{B}_2$ and $\mathscr{B}_0$. So we might add to the above paragraph: (*d*) Notwithstanding (*a*)–(*c*), as α increases for fixed R an oscillatory instability develops when $\alpha = \alpha_0(R)$ if R is so small that $\alpha_0(R) < \alpha_2(R)$. One implication of these ideas is especially important: if a diverging channel with plane walls has semi-angle $\alpha > 4.712/3848$ rad $\approx 0.07°$ then the flow may become unstable because of the divergence before the flow can become unstable owing to the usual Tollmien–Schlichting mechanism of extracting energy from the basic flow at the critical layer. (Note that $\alpha_2(R) \sim 4.712/R$ as $R \to \infty$ along $\mathscr{B}_2$.) Thus the mechanism described by the Orr–Sommerfeld equation may be superseded by the mechanism of instability, which is essentially a *rotation* (corresponding to a θ-shift as exemplified by the eigensolution (3.6*a*) viz. $\lambda = 0$, $\hat{\psi} = G_{0y}$) of the distorted basic Jeffery–Hamel flow, described in this paper if $\alpha \gtrsim 0.07°$. If the channel converges, i.e. $\alpha < 0$, then the mechanism described in this paper is a stabilizing one, so that only the Tollmien–Schlichting mechanism may lead to instability for such values of α.

The laboratory experiments of Sobey (1985) and Sobey & Drazin (1986) show that three-dimensional flow arises from weak side effects at all values of R and from strong instabilities at critical values, although the flow in a wide channel is usually approximately two-dimensional near the centre of the channel at moderate values of R. Therefore the mathematical results of this paper are relevant to real flows, but in applying them the possibility of three-dimensional side effects and instabilities as well as two-dimensional wave instabilities must be borne in mind.

REFERENCES

Allmen, M. J. & Eagles, P. M. 1984 Stability of divergent channel flows: a numerical approach. *Proc. R. Soc. Lond.* A **392**, 359–372.

Brady, J. F. 1984 Fluid development in a porous channel and tube. *Phys. Fluids* **27**, 1061–1067.

Bramley, J. S. & Dennis, S. C. R. 1982 The calculation of eigenvalues for the stationary perturbation of Poiseuille flow. *J. Comp. Phys.* **47**, 179–198.

BUITRAGO, S. E. 1983 Detailed analysis of the higher Jeffery–Hamel solutions. M.Phil. thesis, University of Sussex.

CHERDRON, W., DURST, F. & WHITELAW, J. H. 1978 Asymmetric flows and instabilities in symmetric ducts with sudden expansions. *J. Fluid Mech.* **84**, 13–31.

CLIFFE, K. A. & GREENFIELD, A. C. 1982 Some comments on laminar flow in symmetric two-dimensional channels. *Rep.* TP 939. AERE, Harwell.

DEAN, W. R. & MONTAGNON, P. E. 1949 On the steady motion of viscous liquid in a corner. *Proc. Camb. Phil. Soc.* **45**, 389–394.

DRAZIN, P. G. & REID, W. H. 1981 *Hydrodynamic Stability*. Cambridge University Press.

FRAENKEL, L. E. 1962 Laminar flow in symmetrical channels with slightly curved walls. I. On the Jeffery–Hamel solutions for flow between plane walls. *Proc. R. Soc. Lond.* A**267**, 119–138.

FRAENKEL, L. E. 1963 Laminar flow in symmetrical channels with slightly curved walls. II. An asymptotic series for the stream function. *Proc. R. Soc. Lond.* A**272**, 406–428.

FRAENKEL, L. E. 1973 On a theory of laminar flow in channels of a certain class. *Proc. Camb. Phil. Soc.* **73**, 361–390.

GEORGIOU, G. A. & EAGLES, P. M. 1985 The stability of flows in channels with small wall curvature. *J. Fluid Mech.* **159**, 259–287.

HAMEL, G. 1916 Spiralförmige Bewegungen zäher Flüssigkeiten. *Jahresbericht der Deutschen Math. Vereinigung* **25**, 34–60.

JEFFERY, G. B. 1915 The two-dimensional steady motion of a viscous fluid. *Phil. Mag.* (6) **29**, 455–465.

LUGT, H. J. & SCHWIDERSKI, E. W. 1965 Flows around dihedral angles. I. Eigenmotion and analysis. *Proc. R. Soc. Lond.* A**285**, 382–399.

MOFFATT, H. K. & DUFFY, B. R. 1980 Local similarity solutions and their limitations. *J. Fluid Mech.* **96**, 299–313.

SOBEY, I. J. 1985 Observation of waves during oscillatory channel flow. *J. Fluid Mech.* **151**, 395–426.

SOBEY, I. J. & DRAZIN, P. G. 1986 Bifurcation of two-dimensional channel flows. *J. Fluid Mech.* **171**, 263–287.

STERNBERG, E. & KOITER, W. T. 1958 The wedge under a concentrated couple: a paradox in the two-dimensional theory of elasticity. *Trans. ASME* E: *J. Appl. Mech.* **25**, 575–581.